Project systems theory

Resit exam 2017–2018, Thursday 12 April 2018, 14:00 – 17:00

Problem 1

(4 + 8 = 12 points)

A simple model for the metabolism of alcohol in the body is given by

$$\begin{aligned} \dot{c}_b(t) &= q_b\big(c_l(t) - c_b(t)\big), \\ \dot{c}_l(t) &= q_l\big(c_b(t) - c_l(t)\big) - \phi\big(c_l(t)\big) + u(t), \end{aligned} \tag{1}$$

where $c_b(t)$ and $c_l(t)$ are the concentrations of alcohol in the body and liver, respectively. The intake of alcohol is given by the input $u(t)$ and the function

$$\phi(c_l) = q_{\max}\frac{c_l}{c_0 + c_l} \tag{2}$$

gives the rate at which the liver reduces the alcohol concentration. The constants q_b, q_l, $q_{\max}$, and c_0 are all positive.

a) Let $u(t) = \bar{u}$ be a constant alcohol consumption with $0 < \bar{u} < q_{\max}$. Give the corresponding equilibrium point $(\bar{c}_b, \bar{c}_l)$ of (1).

b) Linearize the system around the equilibrium point given by $(\bar{c}_b, \bar{c}_l)$ and $\bar{u}$. Use the notation $(\bar{c}_b, \bar{c}_l)$ instead of the expressions obtained in a).

Problem 2

(16 points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & -4a & -b & -a \end{bmatrix} x(t), \tag{3}$$

where $a, b \in \mathbb{R}$. Determine the values of a and b for which the system is (asymptotically) stable.

Problem 3

(4 + 12 + 6 = 22 points)

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{4}$$

with state $x(t) \in \mathbb{R}^2$, input $u(t) \in \mathbb{R}$, and where

$$A = \begin{bmatrix} -7 & -4 \\ 4 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}. \tag{5}$$

a) Is the system controllable?

b) Find a nonsingular matrix T and real numbers α_1, α_2 such that

$$T^{-1}AT = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

c) Use the matrix T from problem b) to obtain a state feedback of the form $u = Fx$ such that the closed-loop system matrix $A + BF$ has eigenvalues at -1 and -2.

Problem 4

(3 + 3 + 3 + 3 + 4 + 6 = 22 points)

Consider the system

$$\dot{x}(t) = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 6 & -4 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t), \qquad y(t) = [\,1\ -1\ 1\,]x(t). \tag{6}$$

a) Is the system (asymptotically) stable?

b) Is the system controllable?

c) Is the system stabilizable?

d) Is the system observable?

e) Determine the unobservable subspace and give a basis for this subspace.

For the remainder of this problem, consider the system

$$\dot{x}(t) = \begin{bmatrix} 2 & 2-a & 1-a \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} x(t), \qquad y(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x(t) \tag{7}$$

with $a \in \mathbb{R}$.

f) Determine the values of a for which the system (7) is detectable.

Problem 5

(6 + 12 = 18 points)

Consider the discrete-time system

$$x_{k+1} = Ax_k + Bu_k, \tag{8}$$

with state $x_k \in \mathbb{R}^n$ and input $u_k \in \mathbb{R}^m$.

a) Let the initial condition $x_0 \in \mathbb{R}^n$ as well as the input sequence $\{u_0, u_1, \ldots\}$ be given. Show that the solution of (8) for $k \geq 0$ is given by

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} Bu_i. \tag{9}$$

A discrete-time system (8) is said to be *controllable* if, for every initial condition $x_0 \in \mathbb{R}^n$ and every final state $\bar{x} \in \mathbb{R}^n$, there exists an integer $K > 0$ and an input sequence $\{u_0, u_1, \ldots, u_{K-1}\}$ such that $x_K = \bar{x}$, with x_K the solution at step K as in (9).

b) Prove that (8) is controllable if and only if

$$\operatorname{rank}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n. \tag{10}$$

(10 points free)

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